The Hilbert–Smith conjecture for three-manifolds

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Abstract

We show that every locally compact group which acts faithfully on a connected three-manifold is a Lie group. By known reductions, it suffices to show that there is no faithful action of $\mathbb{Z}_p$ (the p-adic integers) on a connected three-manifold. If $\mathbb{Z}_p$ acts faithfully on M^3, we find an interesting $\mathbb{Z}_p$-invariant open set $U \subseteq M$ with $H_2(U) = \mathbb{Z}$ and analyze the incompressible surfaces in U representing a generator of $H_2(U)$. It turns out that there must be one such incompressible surface, say F, whose isotopy class is fixed by $\mathbb{Z}_p$. An analysis of the resulting homomorphism $\mathbb{Z}_p \to \mathrm{MCG}(F)$ gives the desired contradiction. The approach is local on M.

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1 Introduction

By a *faithful action* of a topological group G on a topological manifold M, we mean a continuous injection $G \to \mathrm{Homeo}(M)$ (where $\mathrm{Homeo}(M)$ has the compact-open topology). A well-known problem is to characterize the locally compact topological groups which can act faithfully on a manifold. In particular, there is the following conjecture, which is a natural generalization of Hilbert's Fifth Problem.

Conjecture 1.1. *If a locally compact topological group G acts faithfully on a connected n-manifold M, then G is a Lie group.*

It is well-known (see, for example, Lee [16] or Tao [39]) that as a consequence of the solution of Hilbert's Fifth Problem (by Gleason [7, 8], Montgomery–Zippin [20, 21], as well as further work by Yamabe [44, 45]), any counterexample to Conjecture 1.1 must contain an embedded copy of $\mathbb{Z}_p$ (the p-adic integers). Thus it is equivalent to consider the following conjecture.

Conjecture 1.2. *There is no faithful action of $\mathbb{Z}_p$ on a connected n-manifold M.*

Conjecture 1.1 also admits the following reformulation, which we hope will help the reader better understand its flavor.

Conjecture 1.3. *Given a connected n-manifold M with metric d and open set $U \subseteq M$, there exists $\epsilon > 0$ such that the subset:*

$$\left\{\phi \in \operatorname{Homeo}(M) \;\middle|\; d(x, \phi(x)) < \epsilon \text{ for all } x \in U\right\} \tag{1.1}$$

of $\operatorname{Homeo}(M)$ *contains no nontrivial compact subgroup.*

For any specific manifold M, Conjectures 1.1–1.3 for M are equivalent. They also have the following simple consequence for *almost periodic* homeomorphisms of M (a homeomorphism is said to be almost periodic if and only if the subgroup of $\operatorname{Homeo}(M)$ it generates has compact closure; see Gottschalk [9] for other equivalent definitions).

Conjecture 1.4. *For every almost periodic homeomorphism f of a connected n-manifold M, there exists $r > 0$ such that f^r is in the image of some homomorphism $\mathbb{R} \to \operatorname{Homeo}(M)$.*

Conjecture 1.1 is known in the cases $n = 1, 2$ (see Montgomery–Zippin [21, pp233,249]). By consideration of $M \times \mathbb{R}$, clearly Conjecture 1.1 in dimension n implies the same in all lower dimensions.

The most popular approach to the conjectures above is via Conjecture 1.2. Yang [46] showed that for any counterexample to Conjecture 1.2, the orbit space $M/\mathbb{Z}_p$ must have cohomological dimension $n + 2$. Conjecture 1.2 has been established for various regularity classes of actions, C^2 actions by Bochner–Montgomery [3], $C^{0,1}$ actions by Repovš–Ščepin [30], $C^{0,\frac{n}{n+2}+\epsilon}$ actions by Maleshich [17], and quasiconformal actions by Martin [18]. In the negative direction, it is known that there *does* exist a continuous decomposition of $\mathbb{R}^n$ into cantor sets for every $n \geq 3$ (see Wilson [43, Theorem 3] and Walsh [42]). By work of Raymond–Williams [29], there are faithful actions of $\mathbb{Z}_p$ on n-dimensional compact metric spaces which achieve the cohomological dimension jump of Yang [46] for every $n \geq 2$.

In this paper, we establish the aforementioned conjectures for $n = 3$.

Theorem 1.5. *There is no faithful action of $\mathbb{Z}_p$ on a connected three-manifold M.*

1.1 Rough outline of the proof of Theorem 1.5

We suppose the existence of a continuous injection $\mathbb{Z}_p \to \operatorname{Homeo}(M)$ and derive a contradiction.

Since $p^k\mathbb{Z}_p \cong \mathbb{Z}_p$, we may replace $\mathbb{Z}_p$ with one of its subgroups $p^k\mathbb{Z}_p$ for any large $k \geq 0$. The subgroups $p^k\mathbb{Z}_p \subseteq \mathbb{Z}_p$ form a neighborhood base of the identity in $\mathbb{Z}_p$; hence by continuity of the action, as $k \to \infty$ these subgroups converge to the identity map on M. By picking a suitable Euclidean chart of M and a suitably large $k \geq 0$, we reduce to the case where M is an open subset of $\mathbb{R}^3$ and the action of $\mathbb{Z}_p$ is very close to the identity.[1]

[1]There are two motivations for this reduction. First, recall that a topological group is NSS ("has no small subgroups") iff there exists an open neighborhood of the identity which contains no nontrivial subgroup. Then a theorem of Yamabe [45, p364 Theorem 3] says that a locally compact topological group is a Lie group iff it is NSS. Thus the relevant property of $\mathbb{Z}_p$ which distinguishes it from a Lie group is the existence of the small subgroups $p^k\mathbb{Z}_p$. Second, recall Newman's theorem [22] which implies that a compact Lie group acting nontrivially on a manifold must have large orbits. In essence, we extending Newman's theorem to the group $\mathbb{Z}_p$ (however the proof will be quite different).

The next step is to produce a compact connected $\mathbb{Z}_p$-invariant subset $Z \subseteq M$ satisfying the following two properties:[2]

1. On a coarse scale, Z looks like a handlebody of genus two.

2. The action of $\mathbb{Z}_p$ on $H^1(Z)$ is nontrivial.

The eventual contradiction will come by combining the first (coarse) property of Z with the second (fine) property of Z. Constructing such a set Z follows a natural strategy: we take the orbit of a closed handlebody and attach the orbit of a small loop connecting two points on the boundary. However, the construction requires checking certain connectedness properties of a number of different orbit sets, and is currently the least transparent part of the proof.

Now we consider an open set U defined roughly as $N_\epsilon(Z) \setminus Z$ (only roughly, since we need to ensure that U is $\mathbb{Z}_p$-invariant and $H_2(U) = \mathbb{Z}$). The set of isotopy classes of incompressible surfaces in U representing a generator of $H_2(U)$ forms a lattice, and this lets us find an incompressible surface F in U which is fixed up to isotopy by $\mathbb{Z}_p$. We think of the surface F as a sort of "approximate boundary" of Z. Even though $\mathbb{Z}_p$ does not act naturally on F itself, we do get a natural homomorphism $\mathbb{Z}_p \to \mathrm{MCG}(F)$ with finite image. The two properties of Z translate into the following two properties of the action of $\mathbb{Z}_p$ on $H_1(F)$:

1. There is a submodule of $H_1(F)$ fixed by $\mathbb{Z}_p$ on which the intersection form is:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.2}$$

2. The action of $\mathbb{Z}_p$ on $H_1(F)$ is nontrivial.

This means we have a cyclic subgroup $\mathbb{Z}/p \subseteq \mathrm{MCG}(F)$ such that $H_1(F)^{\mathbb{Z}/p}$ has a submodule on which the intersection form is given by (1.2). The Nielsen classification of cyclic subgroups of the mapping class group shows that this is impossible. This contradiction completes the proof of Theorem 1.5.

We conclude this introduction with a few additional remarks on the proof. First, the proof is a local argument on M, similar in that respect to the proof of Newman's theorem [22]. Second, our proof works essentially verbatim with any pro-finite group in place of $\mathbb{Z}_p$ (though this is not particularly surprising). Finally, we remark that assuming the action of $\mathbb{Z}_p$ on M is *free* (as is traditional in some approaches to Conjecture 1.2) does not produce any significant simplifications to the argument.

[2] The motivation to consider such a set is to attempt a *dimension reduction* argument. In other words, we would like to conclude that ∂Z is a closed surface with a faithful action of $\mathbb{Z}_p$, and therefore contradicts the (known) case of Conjecture 1.2 with $n = 2$. This, of course, is not possible since Z could *a priori* have wild boundary. We will nevertheless be able to construct a closed surface F which serves as an "approximate boundary" of Z.

1.2 Acknowledgements

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2 The lattice of incompressible surfaces

In this section, we study a natural partial order on the set of incompressible surfaces in a particularly nice class of three-manifolds, which we call "approximate surfaces" (see Definition 2.3). For an approximate surface M, we let $\mathcal{S}(M)$ denote the set of isotopy classes of incompressible surfaces in M which generate $H_2(M)$. The main result of this section (suggested by Ian Agol [1]) is that $\mathcal{S}(M)$ is a lattice (see Lemma 2.18) under its natural partial order. Related ideas may be found in papers of Schultens [34] and Przytycki–Schultens [28] on the contractibility of the Kakimizu complex [13].[3]

Since we will ultimately use the results of this section to study groups of *homeomorphisms*, we need constructions which are functorial with respect to homeomorphisms. To study the properties of these constructions, however, we use methods and results in PL/DIFF three-manifold theory (for example, general position). Thus in this section we will always state explicitly which category (TOP/PL/DIFF) we are working in, since this will change frequently (and there is no straightforward way of working just in a single category). We will, of course, need the key result that every topological three-manifold can be triangulated, as proved by Moise [19] and Bing [2] (see also Hamilton [11] for a modern proof based on the methods of Kirby–Siebenmann [15]; the PL structure is unique, but we do not need this).

By a surface we always mean a closed connected orientable surface.

Lemma 2.1. *Let M be a PL three-manifold, and let F be a bicollared surface in M. Then there exists an isotopy supported in an arbitrarily small neighborhood of F which maps F to a PL surface.*

Proof. Let $\phi : F \times [-1, 1] \to M$ be a bicollar, which we may assume is arbitrarily close to $F = \phi(F \times \{0\})$. Now homeomorphisms of three-manifolds can be approximated by PL homeomorphisms. Thus let us pick a PL structure on F and apply [2, p62 Theorem 8] to get $\phi' : F \times [-1, 1] \to M$ which is PL in the range $F \times (0, 1)$, and coincides with ϕ on $F \times [-1, 0]$. Now using the bicollar ϕ' we can easily construct an isotopy of M which sends $F = \phi'(F \times \{0\})$ to $\phi'(F \times \{\frac{1}{2}\})$, which is PL. $\square$

Lemma 2.2. *Let M be an irreducible orientable TOP (resp. PL) three-manifold, and let F_1, F_2 be two bicollared (resp. PL) π_1-injective surfaces in M. If F_1, F_2 are homotopic, then there is a compactly supported (resp. PL) isotopy of M which sends F_1 to F_2.*

[3] The Kakimizu complex is a sort of "$\mathbb{Z}$-equivariant order complex" of $\mathcal{S}(\widetilde{\mathbb{S}^3 \setminus K})$, where $\widetilde{\mathbb{S}^3 \setminus K}$ denotes the infinite cyclic cover of the knot complement. Even though technically $\widetilde{\mathbb{S}^3 \setminus K}$ is not an approximate surface under our definition, it is easy to give a modified definition (allowing manifolds with boundary) to which the methods of this section would apply.

Proof. Waldhausen [41, p76 Corollary 5.5] proves this in the PL category if M is compact with boundary. It is clear that this implies our PL statement, since the given homotopy will be supported in a compact region of M.

For the TOP category, first pick a PL structure on M, and use Lemma 2.1 to straighten F_1, F_2 by a compactly supported isotopy. Now use the PL version of this lemma. $\square$

Definition 2.3. An *approximate surface* is an irreducible orientable three-manifold with exactly two ends and $H_2 \cong \mathbb{Z}$. For example, $\Sigma_g \times \mathbb{R}$ is an approximate surface for $g \geq 1$.

Lemma 2.4. *Let M be an approximate surface. For an embedded surface $F \subseteq M$, the following are equivalent:*

1. *F is nonzero in $H_2(M)$.*
2. *F separates the two ends of M.*
3. *F generates $H_2(M)$.*

Proof. (1) $\implies$ (2). A path from one end to the other gives a non-torsion class in $H_1^{\mathrm{lf}}(M)$. Thus its Poincaré dual is a non-torsion class in $H^2(M)$, and thus defines a nonzero map $H_2(M) \to \mathbb{Z}$. Since F is nonzero in $H_2(M) = \mathbb{Z}$, every such path must therefore intersect F.

(2) $\implies$ (3). If F separates the two ends, then there is a path from one end to the other which intersects F exactly once. Thus the Poincaré dual of the class of this path in $H_1^{\mathrm{lf}}(M)$ evaluates to 1 on $F \in H_2(M)$. Thus F represents a primitive element of $H_2(M) = \mathbb{Z}$, and thus generates it.

(3) $\implies$ (1). Trivial. $\square$

Definition 2.5. For a TOP approximate surface M, let $\mathcal{S}_{\mathrm{TOP}}(M)$ be the set of bicollared π_1-injective embedded surfaces in M generating $H_2(M)$, modulo homotopy.

Definition 2.6. For a PL approximate surface M, let $\mathcal{S}_{\mathrm{PL}}(M)$ be the set of PL π_1-injective embedded surfaces in M generating $H_2(M)$, modulo homotopy.

Remark 2.7. $\mathcal{S}_{\mathrm{PL}}(M)$ is always nonempty, since we can pick a PL embedded surface representing a generator of $H_2(M)$ and then take some maximal compression, which will be π_1-injective by the loop theorem.

Definition 2.8. A *marked approximate surface* is an approximate surface along with a labelling of the ends with $\pm$. For an embedded surface $F \subseteq M$ separating the two ends, let $(M \setminus F)_\pm$ denote the two connected components of $M \setminus F$ (the subscripts corresponding to the labelling of the ends).

Definition 2.9. Let M be a marked approximate surface. For two disjoint embedded surfaces $F_1, F_2 \subseteq M$ separating the two ends of M, we say $F_1 \leq F_2$ iff $F_1 \subseteq (M \setminus F_2)_-$. Define a relation $\leq$ on $\mathcal{S}_{\mathrm{TOP}}(M)$ (resp. $\mathcal{S}_{\mathrm{PL}}(M)$) by declaring that $\mathfrak{F}_1 \leq \mathfrak{F}_2$ if and only if $\mathfrak{F}_1, \mathfrak{F}_2$ have disjoint embedded representatives F_1, F_2 with $F_1 \leq F_2$.

Lemma 2.10. *Let M be a PL marked approximate surface. Then the natural map ψ : $\mathcal{S}_{\mathrm{PL}}(M) \to \mathcal{S}_{\mathrm{TOP}}(M)$ is a bijection satisfying $\mathfrak{F}_1 \leq \mathfrak{F}_2 \iff \psi(\mathfrak{F}_1) \leq \psi(\mathfrak{F}_2)$.*

Proof. The natural map $(\mathcal{S}_{\mathrm{PL}}(M), \leq) \to (\mathcal{S}_{\mathrm{TOP}}(M), \leq)$ is clearly injective, and by Lemma 2.1 it is surjective.

If $\mathfrak{F}_1, \mathfrak{F}_2$ have bicollared representatives F_1, F_2 with $F_1 \leq F_2$, then by Lemma 2.1 they can be straightened preserving the relation $F_1 \leq F_2$. The other direction is obvious, since PL surfaces have a bicollar. □

Having established that $\mathcal{S}_{\mathrm{PL}}(M)$ and $\mathcal{S}_{\mathrm{TOP}}(M)$ are naturally isomorphic, we henceforth use the notation $\mathcal{S}(M)$ for both.

Lemma 2.11. *Let M be a PL approximate surface. Suppose F is a PL embedded surface in M separating the two ends of M, and suppose γ is a PL arc from one end of M to the other which intersects F transversally in exactly one point. Denote by $\pi_1(M,\gamma)$ one of the groups $\{\pi_1(M,p)\}_{p\in\gamma}$ (they are all* naturally *isomorphic given the path γ).*[4] *Then for any surface $G \subseteq M$ homotopic to F, there is a canonical map $\pi_1(G) \to \pi_1(M,\gamma)$ (defined up to inner automorphism of the domain).*

Proof. Assume that G intersects γ transversally. Let us call two intersections of γ with G *equivalent* iff there is a path between them on G which, when spliced with the path between them on γ, becomes null-homotopic in M (this is an equivalence relation). Note that during a general position homotopy of G, the mod 2 cardinalities of the equivalence classes of intersections with γ remain the same. Thus since G is homotopic to F and $\#(F \cap \gamma) = 1$, there is a unique equivalence class of intersection $G \cap \gamma$ of odd cardinality. Picking any one of these points as basepoint on G and on γ gives the same map $\pi_1(G) \to \pi_1(M,\gamma)$ up to inner automorphism of the domain. This map is clearly constant under homotopy of G. □

Lemma 2.12. *Let M be a PL approximate surface. Let F be a PL π_1-injective surface in M separating the two ends of M. Then any homotopy of F to itself induces the trivial element of* $\mathrm{MCG}(F)$.

Proof. Pick a PL arc γ from one end of M to the other which intersects F transversally exactly once. By Lemma 2.11, we get a canonical map $\pi_1(F) \to \pi_1(M,\gamma)$ which is constant as we move F by homotopy. Since this map is injective, we know $\pi_1(F)$ up to inner automorphism as a subgroup of $\pi_1(M,\gamma)$. □

Lemma 2.13. *Let M be a marked approximate surface. Then the pair $(\mathcal{S}(M), \leq)$ is a partially-ordered set. That is, for all $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \in \mathcal{S}(M)$ we have:*

1. *(reflexivity) $\mathfrak{F}_1 \leq \mathfrak{F}_1$.*
2. *(antisymmetry) $\mathfrak{F}_1 \leq \mathfrak{F}_2 \leq \mathfrak{F}_1 \implies \mathfrak{F}_1 = \mathfrak{F}_2$.*
3. *(transitivity) $\mathfrak{F}_1 \leq \mathfrak{F}_2 \leq \mathfrak{F}_3 \implies \mathfrak{F}_1 \leq \mathfrak{F}_3$.*

Proof. By Lemma 2.10, it suffices to work in the PL category.

Reflexivity is obvious.

[4] A categorical way of phrasing this is as follows. Let $\boldsymbol{\gamma}$ denote the category whose objects are points $p \in \gamma$, with a single morphism $p \to p'$ for all $p, p' \in \gamma$. The fundamental group is a functor $\pi_1(M, \cdot) : \boldsymbol{\gamma} \to \mathfrak{Groups}$, where the morphism $p \to p'$ is sent to the isomorphism $\pi_1(M,p) \to \pi_1(M,p')$ given by the path from p to p' along γ. Then $\pi_1(M,\gamma)$ is defined as the limit/colimit of this functor.

For transitivity, suppose $\mathfrak{F}_1 \leq \mathfrak{F}_2 \leq \mathfrak{F}_3$. Then we have representatives F_1, F_2, F_2', F_3' such that $F_1 \leq F_2$ and $F_2' \leq F_3'$. By Lemma 2.2, there is an isotopy from F_2 to F_2'. Applying this isotopy to F_1 produces $F_1' \leq F_2' \leq F_3'$.

For antisymmetry, suppose $\mathfrak{F}_1 \leq \mathfrak{F}_2$ and $\mathfrak{F}_2 \leq \mathfrak{F}_1$. Pick PL representatives $F_1 \leq F_2 \leq F_1' \leq F_2'$ (this is possible by the argument used for transitivity). Now pick a PL arc γ from one end of M to the other which intersects each of F_1, F_1', F_2, F_2' exactly once. Since the maps $\pi_1(F_1), \pi_1(F_1'), \pi_1(F_2), \pi_1(F_2') \to \pi_1(M, \gamma)$ are injective, we will identify each of the groups on the left with its image in $\pi_1(M, \gamma)$. By Lemma 2.11, we have $\pi_1(F_i) = \pi_1(F_i')$ under this identification.

By van Kampen's theorem we have:

$$\pi_1(M, \gamma) = \pi_1((M \setminus F_2)_-, \gamma) \underset{\pi_1(F_2)}{*} \pi_1((M \setminus F_2)_+, \gamma) \tag{2.1}$$

Since $F_1 \leq F_2 \leq F_1'$, we have $\pi_1(F_1) \subseteq \pi_1((M \setminus F_2)_-, \gamma)$ and $\pi_1(F_1') \subseteq \pi_1((M \setminus F_2)_+, \gamma)$. Since $\pi_1(F_1) = \pi_1(F_1')$, we have $\pi_1(F_1) \subseteq \pi_1((M \setminus F_2)_-, \gamma) \cap \pi_1((M \setminus F_2)_+, \gamma)$. It follows from the properties of the amalgamated product (2.1) that this intersection is just $\pi_1(F_2)$. Thus we have $\pi_1(F_1) \subseteq \pi_1(F_2)$. A symmetric argument shows the reverse inclusion, so we have $\pi_1(F_1) = \pi_1(F_2)$. Now an easy obstruction theory argument shows F_1 and F_2 are homotopic (using the fact that $\pi_2(M) = 0$ by the sphere theorem [27]). Thus $\mathfrak{F}_1 = \mathfrak{F}_2$. □

Lemma 2.14. *Let M be a marked approximate surface. Let F be a bicollared embedded π_1-injective surface. Then the natural map $\psi : \mathcal{S}((M \setminus F)_-) \to \mathcal{S}(M)$ gives a bijection of $\mathcal{S}((M \setminus F)_-)$ with the set $\{\mathfrak{G} \in \mathcal{S}(M) \mid \mathfrak{G} \leq [F]\}$. Furthermore, this bijection satisfies $\mathfrak{G}_1 \leq \mathfrak{G}_2 \iff \psi(\mathfrak{G}_1) \leq \psi(\mathfrak{G}_2)$.*

Proof. By Lemma 2.10, it suffices to work in the PL category.

Certainly the image of ψ is contained in $\{\mathfrak{G} \in \mathcal{S}(M) \mid \mathfrak{G} \leq [F]\}$. If $\mathfrak{G} \leq [F]$, then there are representatives $G' \leq F'$. By Lemma 2.2, there is an isotopy sending F' to F. Applying this isotopy to G' gives a representative $G \leq F$. Thus the image of ψ is exactly $\{\mathfrak{G} \in \mathcal{S}(M) \mid \mathfrak{G} \leq [F]\}$. To prove that ψ is injective, suppose $G_1, G_2 \leq F$ are two π_1-injective embedded surfaces which are homotopic in M. Pick an arc γ from one end of M to the other which intersects G_1, F exactly once. Since G_1, G_2 are homotopic, Lemma 2.11 gives canonical maps $\pi_1(G_1), \pi_1(G_2) \to \pi_1(M, \gamma)$ with the same image. Now these both factor through $\pi_1((M \setminus F)_-) \to \pi_1(M, \gamma)$, which is injective since $\pi_1(F) \to \pi_1((M \setminus F)_\pm)$ are injective. Thus $\pi_1(G_1), \pi_1(G_2) \to \pi_1((M \setminus F)_-)$ have the same image, so the same obstruction theory argument used in the proof of Lemma 2.13 implies that G_1, G_2 are homotopic in $(M \setminus F)_-$. Thus ψ is injective.

Now it remains to show that ψ preserves $\leq$. The only nontrivial direction is to show that $\psi(\mathfrak{G}_1) \leq \psi(\mathfrak{G}_2) \implies \mathfrak{G}_1 \leq \mathfrak{G}_2$. If $\psi(\mathfrak{G}_1) \leq \psi(\mathfrak{G}_2)$, then we have representatives $G_1' \leq G_2' \leq F'$ (by the argument for transitivity in Lemma 2.13). Now by Lemma 2.2 there is an isotopy from F' to F, and applying this isotopy to G_1', G_2', we get $G_1 \leq G_2 \leq F$. Since ψ is injective, we have $[G_i] = \mathfrak{G}_i$ in $\mathcal{S}((M \setminus F)_-)$. Thus $\mathfrak{G}_1 \leq \mathfrak{G}_2$. □

Lemma 2.15. *Let M be a marked approximate surface. Let $F_1 \leq F_2$ be two bicollared embedded π_1-injective surfaces. Then the natural map $\psi : \mathcal{S}((M \setminus F_1)_+ \cap (M \setminus F_2)_-) \to \mathcal{S}(M)$ gives a bijection of $\mathcal{S}((M \setminus F_1)_+ \cap (M \setminus F_2)_-)$ with $\{\mathfrak{G} \in \mathcal{S}(M) \mid [F_1] \leq \mathfrak{G} \leq [F_2]\}$. Furthermore, this bijection satisfies $\mathfrak{G}_1 \leq \mathfrak{G}_2 \iff \psi(\mathfrak{G}_1) \leq \psi(\mathfrak{G}_2)$.*

Proof. This is just two applications of Lemma 2.14. $\square$

Lemma 2.16. *Let M be a PL marked approximate surface. Let $G \subseteq M$ be any PL embedded surface (not necessarily π_1-injective) separating the two ends of M. Then there exists a class $\mathfrak{G} \in \mathcal{S}(M)$ such that for all $\mathfrak{F} \in \mathcal{S}(M)$ with a representative $F \leq G$ (resp. $F \geq G$), we have $\mathfrak{F} \leq \mathfrak{G}$ (resp. $\mathfrak{F} \geq \mathfrak{G}$).*

Proof. As long as G is compressible, we can perform the following operation. Do some compression on G (this may disconnect G), and pick one of the resulting connected components which is nonzero in $H_2(M)$ to keep. This operation does not destroy the property that G can be isotoped to lie in $(M \setminus F)_-$ (resp. $(M \setminus F)_+$) for an incompressible surface F. Since each step decreases the genus of G, we eventually reach an incompressible surface, which by the loop theorem is π_1-injective, and thus defines a class $\mathfrak{G} \in \mathcal{S}(M)$. $\square$

Lemma 2.17. *Let M be a marked approximate surface. Let $A \subseteq \mathcal{S}(M)$ be a finite set. Then there exist elements $\mathfrak{A}_-, \mathfrak{A}_+ \in \mathcal{S}(M)$ such that $\mathfrak{A}_- \leq \mathfrak{A} \leq \mathfrak{A}_+$ for all $\mathfrak{A} \in A$.*

Proof. By Lemma 2.10, it suffices to work in the PL category.

Pick PL transverse representatives $A_1, \ldots, A_n$ of all the surfaces in A. Now Lemma 2.16 applied to $\partial((M \setminus A_1)_- \cap \cdots \cap (M \setminus A_n)_-)$ and $\partial((M \setminus A_1)_+ \cap \cdots \cap (M \setminus A_n)_+)$ gives the desired classes $\mathfrak{A}_-, \mathfrak{A}_+ \in \mathcal{S}(M)$. $\square$

Lemma 2.18 (suggested by Ian Agol [1])**.** *Let M be a marked approximate surface. Then the partially-ordered set $(\mathcal{S}(M), \leq)$ is a lattice. That is, for $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{S}(M)$, the following set has a least element:*

$$X(\mathfrak{F}_1, \mathfrak{F}_2) = \left\{\mathfrak{H} \in \mathcal{S}(M) \;\middle|\; \mathfrak{F}_1, \mathfrak{F}_2 \leq \mathfrak{H}\right\} \tag{2.2}$$

(and the same holds with $\leq$ replaced with $\geq$).

Proof. By Lemma 2.10, it suffices to work in the PL category.

First, let us deal with the case where M has tame ends, that is M is the interior of a compact PL manifold with boundary $(M, \partial M)$. Equip $(M, \partial M)$ with a smooth structure and smooth Riemannian metric so that the boundary is convex.

Now let us recall some facts about area minimizing surfaces. The standard existence theory of Schoen–Yau [33] and Sacks–Uhlenbeck [31, 32] gives existence of area minimizing maps in any homotopy class of π_1-injective surfaces. By Osserman [26, 25] and Gulliver [10], such area minimizing maps are immersions. Now Freedman–Hass–Scott [6] have proved theorems to the effect that minimal area representatives of π_1-injective surfaces in irreducible three-manifolds have as few intersections as possible given their homotopy class. Though they state their main results only for the case of closed irreducible three-manifolds, they remark [6, p634 §7] that their results remain true if one allows convex boundary.

Suppose $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{S}(M)$; let us show that $X(\mathfrak{F}_1, \mathfrak{F}_2)$ has a least element. Pick minimal area representatives F_1, F_2 in the homotopy classes of $\mathfrak{F}_1, \mathfrak{F}_2$. By [6, p626 Theorem 5.1], F_1, F_2 are embedded. Consider the piecewise smooth surface $F := \partial((M \setminus F_1)_+ \cap (M \setminus F_2)_+)$. Note that F_1, F_2 can be isotoped to lie in $(M \setminus F)_-$. Furthermore, if $\mathfrak{G} \in X(\mathfrak{F}_1, \mathfrak{F}_2)$ (and $\mathfrak{G} \neq \mathfrak{F}_1, \mathfrak{F}_2$), then $\mathfrak{G}$ has a least area representative G, which by [6, p630 Theorem 6.2] is disjoint from F_1 and F_2. Thus $F_1, F_2 \leq G$, so $F \leq G$. Thus we may apply Lemma 2.16 to

F and produce an element $\mathfrak{F} \in \mathcal{S}(M)$ which a least element of $X(\mathfrak{F}_1, \mathfrak{F}_2)$. This finishes the case of M with tame ends.

Now let us treat the case of general M. Consider the following sets:

$$X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G}) = \{\mathfrak{H} \in \mathcal{S}(M) \mid \mathfrak{F}_1, \mathfrak{F}_2 \leq \mathfrak{H} \leq \mathfrak{G}\} \subseteq X(\mathfrak{F}_1, \mathfrak{F}_2) \tag{2.3}$$

We claim that it suffices to show that if $\mathfrak{G} \in X(\mathfrak{F}_1, \mathfrak{F}_2)$, then $X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G})$ has a least element. To see this, we argue as follows. If $\mathfrak{G}_1, \mathfrak{G}_2 \in X(\mathfrak{F}_1, \mathfrak{F}_2)$ and $\mathfrak{G}_1 \leq \mathfrak{G}_2$, then it is easy to see that the natural inclusion $X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G}_1) \to X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G}_2)$ sends the least element of the domain to the least element of the target (assuming both sets have least elements). Since for every $\mathfrak{G}_1, \mathfrak{G}_2$, there exists $\mathfrak{G}_3$ greater than both (by Lemma 2.17), we see that the least elements of all the sets $\{X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G})\}_{\mathfrak{G} \in X(\mathfrak{F}_1, \mathfrak{F}_2)}$ are actually the same element $\mathfrak{F} \in X(\mathfrak{F}_1, \mathfrak{F}_2)$. We claim that this $\mathfrak{F}$ is a least element of $X(\mathfrak{F}_1, \mathfrak{F}_2)$. This is trivial: if $\mathfrak{G} \in X(\mathfrak{F}_1, \mathfrak{F}_2)$, then $\mathfrak{F}$ is a least element of $X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G})$, so *a fortiori* $\mathfrak{F} \leq \mathfrak{G}$. Thus it suffices to show that each set $X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G})$ has a least element.

Let us show that $X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G})$ has a least element. By Lemma 2.17 there exists $\mathfrak{F}_0 \in \mathcal{S}(M)$ so that $\mathfrak{F}_0 \leq \mathfrak{F}_1, \mathfrak{F}_2$. Now pick PL representatives F_0, G of $\mathfrak{F}_0, \mathfrak{G}$ with $F_0 \leq G$. Let $M_0 = (M \setminus F_0)_+ \cap (M \setminus G)_-$, which is a marked approximate surface with tame ends. Thus by the case dealt with earlier, $\mathcal{S}(M_0)$ is a lattice, so $\mathfrak{F}_1, \mathfrak{F}_2$ have a least upper bound in $\mathcal{S}(M_0)$. By Lemma 2.15, $\mathcal{S}(M_0) \to \mathcal{S}(M)$ is an isomorphism onto the subset $\{\mathfrak{F} \in \mathcal{S}(M) \mid \mathfrak{F}_0 \leq \mathfrak{F} \leq \mathfrak{G}\}$. Thus we get the desired least element of $X(\mathfrak{F}_1, \mathfrak{F}_2; \mathfrak{G})$. □

Remark 2.19. It should be possible to prove an existence result for minimal surfaces in *any* DIFF approximate surface (with appropriate choice of Riemannian metric). In that case, the argument used for tame approximate surfaces would apply in general (the results of Freedman–Hass–Scott [6] extend as long as one has existence of minimal area representatives of π_1-injective surfaces in M).

Remark 2.20. Jaco–Rubinstein [12] have developed a theory of *normal surfaces* in triangulated three-manifolds analogous to that of minimal surfaces in smooth three-manifolds with a Riemannian metric. In particular, they have results analogous to those of Freedman–Hass–Scott [6]. Thus it is likely that we could eliminate entirely the use of the smooth category and results in minimal surface theory for the proof of Lemma 2.18.

3 Tools applicable to arbitrary open/closed subsets of manifolds

In our study of a hypothetical action of $\mathbb{Z}_p$ by homeomorphisms on a three-manifold, it will be essential to study certain properties of orbit sets (for example, their homology). However, we do not have the luxury of assuming such sets are at all well behaved; the most we can hope for is that we will be able to construct $\mathbb{Z}_p$-invariant sets which are either open or closed. Nevertheless, Čech cohomology is still a reasonable object in such situations. The purpose of this section is to develop an elementary theory centered on Alexander duality for arbitrary open and closed subsets of a manifold. We use these tools in the proof of Theorem 1.5, specifically in Sections 4.3–4.4 to construct the set Z and to study its properties.

In this and subsequent sections, we always take homology and cohomology with integer coefficients. We denote by H_* and H^* singular homology and cohomology, and we let $\check{H}^*$ denote Čech cohomology.

3.1 Čech cohomology

Lemma 3.1. *For a compact subset X of a manifold, the natural map below is an isomorphism:*

$$\varinjlim_{\substack{U \supseteq X \\ U \text{ open}}} H^*(U) \xrightarrow{\sim} \check{H}^*(X) \tag{3.1}$$

Proof. This is due to Steenrod [37]; see also Spanier [36, p419], the key fact being that Čech cohomology satisfies the "continuity axiom". □

Lemma 3.2. *If X and Y are two compact subsets of a manifold, then there is a (Mayer–Vietoris) long exact sequence:*

$$\cdots \to \check{H}^*(X \cup Y) \to \check{H}^*(X) \oplus \check{H}^*(Y) \to \check{H}^*(X \cap Y) \to \check{H}^{*+1}(X \cup Y) \to \cdots \tag{3.2}$$

Proof. For arbitrary open neighborhoods $U \supseteq X$ and $V \supseteq Y$, we have a Mayer–Vietoris sequence of singular cohomology for U and V. Taking the direct limit over all U and V gives a sequence of the form (3.2) by Lemma 3.1, and it is exact since the direct limit is an exact functor. □

3.2 Alexander duality

Alexander duality for subsets of $\mathbb{S}^n$ is most naturally stated using reduced homology and cohomology, which we denote by $\tilde{H}$.

Lemma 3.3. *Let $X \subseteq \mathbb{S}^n$ be a compact set. Then:*

$$\tilde{\check{H}}^*(X) = \tilde{H}_{n-1-*}(\mathbb{S}^n \setminus X) \tag{3.3}$$

Proof. If $U \subseteq \mathbb{S}^n$ is a smooth n-dimensional submanifold with boundary, then it is well-known that there is a natural isomorphism $\tilde{H}^*(U) \xrightarrow{\sim} \tilde{H}_{n-1-*}(\mathbb{S}^n \setminus U)$. Furthermore, for $U \supseteq V$ the following diagram commutes:

$$\begin{array}{ccc} \tilde{H}^*(U) & \xrightarrow{\sim} & \tilde{H}_{n-1-*}(\mathbb{S}^n \setminus U) \\ \downarrow & & \downarrow \\ \tilde{H}^*(V) & \xrightarrow{\sim} & \tilde{H}_{n-1-*}(\mathbb{S}^n \setminus V) \end{array} \tag{3.4}$$

Now consider the direct limit over all such open sets $U \supseteq X$ to get an isomorphism of the direct limits. This family of sets U forms a final system of neighborhoods of X, so by Lemma 3.1 the direct limit on the left is $\tilde{\check{H}}^*(X)$. The direct limit on the right is $\tilde{H}_{n-1-*}(\mathbb{S}^n \setminus X)$ by elementary properties of singular homology. □

3.3 The plus operation

If we have a bounded subset $X \subseteq \mathbb{R}^3$, we want to be able to consider "X union all of the bounded connected components of $\mathbb{R}^3 \setminus X$" (we think of this operation as a way to simplify the set X, which could be very wild). The following definition makes this precise.

Definition 3.4. For a bounded set $X \subseteq \mathbb{R}^3$, we define:

$$X^+ = \bigcap_{\substack{\text{open } U \supseteq X \\ U \text{ bounded} \\ H_2(U)=0}} U \tag{3.5}$$

By Lemma 3.3, for bounded open sets $U \subseteq X$, we have $H_2(U) = 0 \iff \check{H}^0(\mathbb{R}^3 \setminus U) = \mathbb{Z}$. With this reformulation, it is easy to see that if U and V both appear in the intersection, then so does $U \cap V$. Let us call $X \mapsto X^+$ the *plus operation*.

Lemma 3.5. *The plus operation satisfies the following properties:*
i. $X \subseteq Y \implies X^+ \subseteq Y^+$.
ii. $X^{++} = X^+$.
iii. If X *is closed and bounded, then* $\mathbb{R}^3 \setminus X^+$ *is the unbounded component of* $\mathbb{R}^3 \setminus X$.
iv. If X *is closed and bounded, then* $X = X^+ \iff \check{H}^2(X) = 0 \iff \mathbb{R}^3 \setminus X$ *is connected.*

Proof. Both (i) and (ii) are immediate from the definition.

For (iii), argue as follows. Let V denote the unbounded component of $\mathbb{R}^3 \setminus X$. As V is open and connected, it has an exhaustion by closed connected subsets $V = \bigcup_{i=1}^{\infty} V_i$, where we may assume that $V_1 \subseteq V_2 \subseteq \cdots$ and $\mathbb{R}^3 \setminus V_i$ is bounded for all i. By Lemma 3.3, $H_2(\mathbb{R}^3 \setminus V_i) = 0$, and thus $X^+ \subseteq \bigcap_{i=1}^{\infty}(\mathbb{R}^3 \setminus V_i) = \mathbb{R}^3 \setminus V$. It remains to show the reverse inclusion, namely that if W is a bounded component of $\mathbb{R}^3 \setminus X$, then $W \subseteq X^+$. Suppose $U \supseteq X$ is open, bounded, and $H_2(U) = 0$. Then $W \setminus U$ is a bounded connected component of $\mathbb{R}^3 \setminus U$. However the latter set is connected and unbounded, so we must have $W \setminus U = \varnothing$, that is $U \supseteq W$.

For (iv), argue as follows. Lemma 3.3 gives $\check{H}^2(X) = 0 \iff \mathbb{R}^3 \setminus X$ is connected. By (iii), we have $X = X^+ \iff \mathbb{R}^3 \setminus X$ is connected. □

Lemma 3.6. *Suppose* $X \subseteq \mathbb{R}^3$ *is closed and* $X = X^+$. *If* $\{U_\alpha\}_{\alpha \in A}$ *is a final collection of open sets containing* X, *then so is* $\{U_\alpha^+\}_{\alpha \in A}$.

Proof. Since $\{U_\alpha\}_{\alpha \in A}$ is final, to show that $\{U_\alpha^+\}_{\alpha \in A}$ is final it suffices to show that for every $\alpha \in A$, there exists $\beta \in A$ such that $U_\beta^+ \subseteq U_\alpha$.

Thus let us suppose $\alpha \in A$ is given. As in the proof of Lemma 3.5(iii), there exists an exhaustion $\mathbb{R}^3 \setminus X = \bigcup_{i=1}^{\infty} V_i$, where we may assume $V_1 \subseteq V_2 \subseteq \cdots$ and $\mathbb{R}^3 \setminus V_i$ is bounded for all i. Then for sufficiently large i, we have $\mathbb{R}^3 \setminus V_i \subseteq U_\alpha$. On the other hand, for every i there exists β such that $U_\beta \subseteq \mathbb{R}^3 \setminus V_i$. But now $U_\beta^+ \subseteq (\mathbb{R}^3 \setminus V_i)^+ = \mathbb{R}^3 \setminus V_i \subseteq U_\alpha$ as needed. □

Remark 3.7. If M is Hausdorff and carries a continuous action of $\mathbb{Z}_p$, then every orbit is either a finite discrete set (if the stabilizer is $p^k\mathbb{Z}_p$) or a cantor set (if the stabilizer is trivial). In particular, every orbit has covering dimension zero. It follows that if $A \subseteq M$ is any finite set, then $\mathbb{Z}_p A$ has covering dimension zero, as does any subset thereof.

Lemma 3.8. *If M is a manifold of dimension $n \geq 2$ and $X \subseteq M$ is closed with covering dimension zero, then the map $H_0(M \setminus X) \to H_0(M)$ is an isomorphism.*

Proof. First, let us deal with the case $M = \mathbb{R}^n$. Let $\bar{X} \subseteq \mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$ be $X \cup \{\infty\}$ (which is compact). Given an open covering $X = \bigcup_{i=1}^n U_i$ of dimension zero, we can construct an open covering of $\bar{X}$ of dimension zero by using $\{U_i\}_{U_i \subseteq B(\mathbf{0},N)}$ and the set $\{\infty\} \cup \bigcup_{U_i \not\subseteq B(\mathbf{0},N)} U_i$. As $N \to \infty$ and the covering $\{U_i\}$ gets arbitrarily fine, we get an arbitrarily fine cover of $\bar{X}$ with dimension zero. Thus $\bar{X}$ has covering dimension zero, so $\check{H}^i(\bar{X}) = 0$ for $i > 0$. Thus by Lemma 3.3, $H_0(\mathbb{R}^n \setminus X) = H_0(\mathbb{S}^n \setminus \bar{X}) = \mathbb{Z}$ (since $n \geq 2$), which is sufficient.

Now let us deal with the case of general M. It suffices to show that $H^0(M) \to H^0(M \setminus X)$ is an isomorphism (recall that H^0 is just the group of locally constant maps to $\mathbb{Z}$). Now consider any $p \in M$, and pick an open neighborhood $p \in U \cong \mathbb{R}^n$. Since X has covering dimension zero, so does $U \cap X$. Thus by the case dealt with above, $U \setminus X$ is connected and nonempty. Thus any locally constant function $M \setminus X \to \mathbb{Z}$ can be extended uniquely to a locally constant function $M \to \mathbb{Z}$, as needed. □

4 Nonexistence of faithful actions of $\mathbb{Z}_p$ on three-manifolds

Proof of Theorem 1.5. Let M be a connected three-manifold and suppose there is a continuous injection of topological groups $\mathbb{Z}_p \to \mathrm{Homeo}(M)$.

4.1 Step 1: Reduction to a local problem in $\mathbb{R}^3$

Let $B(r)$ denote the open ball of radius r centered at the origin $\mathbf{0}$ in $\mathbb{R}^3$. Fix a small positive number $\eta = 2^{-10}$. In this section, we reduce to the case where M is an open subset of $\mathbb{R}^3$ such that:

i. $B(1) \subseteq M \subseteq B(1+\eta)$.
ii. $d_{\mathbb{R}^3}(x, \alpha x) \leq \eta$ for all $x \in M$ and $\alpha \in \mathbb{Z}_p$.
iii. No subgroup $p^k\mathbb{Z}_p$ fixes an open neighborhood of $\mathbf{0}$.

Definition 4.1. We say that an action of a group G on a topological space X is *locally of finite order* at a point $x \in X$ if and only if some subgroup of finite index $G' \leq G$ fixes an open neighborhood of x. The action is *(globally) of finite order* if and only if some subgroup of finite index $G' \leq G$ fixes all of X.

Newman's theorem [22, p6 Theorem 2] (see also Dress [5, p204 Theorem 1] or Smith [35]) implies that on a connected manifold, a group action which is (everywhere) locally of finite order is globally of finite order. Thus if $\mathbb{Z}_p \to \mathrm{Homeo}(M)$ is injective, then there exists a point $m \in M$ where the action is not locally of finite order. In other words, no subgroup $p^k\mathbb{Z}_p$ fixes an open neighborhood of m.

Now fix some homeomorphism between $B(3) \subseteq \mathbb{R}^3$ and an open set in M containing m, such that $\mathbf{0}$ is identified with m. Note that (by continuity of the map $\mathbb{Z}_p \to \mathrm{Homeo}(M)$) for sufficiently large k, we have $p^k\mathbb{Z}_pB(2) \subseteq B(3)$ and $d_{\mathbb{R}^3}(x, \alpha x) \leq \eta$ for all $x \in B(2)$ and $\alpha \in p^k\mathbb{Z}_p$. Thus the action of $p^k\mathbb{Z}_p$ (which is isomorphic to $\mathbb{Z}_p$) on $M' := p^k\mathbb{Z}_pB(1)$ satisfies (i), (ii), and (iii) above. Thus it suffices to replace $(M, \mathbb{Z}_p)$ with $(M', p^k\mathbb{Z}_p)$ and derive a contradiction.

Henceforth we assume that M is an open subset of $\mathbb{R}^3$ satisfying (i), (ii), and (iii) above.

4.2 Step 2: An invariant metric and the plus operation in M

Equip M with the following $\mathbb{Z}_p$-invariant metric, which induces the same topology as $d_{\mathbb{R}^3}$:

$$d_{\text{inv}}(x,y) := \int_{\mathbb{Z}_p} d_{\mathbb{R}^3}(\alpha x, \alpha y)\, d\mu_{\text{Harr}}(\alpha) \tag{4.1}$$

We will never mention d_{inv} again explicitly, but we will often write N_ϵ^{inv} for the open ϵ-neighborhood in M with respect to d_{inv}. Note that if $X \subseteq M$ is $\mathbb{Z}_p$-invariant, then so is $N_\epsilon^{\text{inv}}(X)$.

Lemma 4.2. *If $X \subseteq B(1) \subseteq M$, then $X^+ \subseteq B(1) \subseteq M$. If in addition X is $\mathbb{Z}_p$-invariant, then so is X^+.*

Proof. The first statement is clear since $B(1)$ appears in the intersection (3.5) defining X^+.

Now suppose that in addition X is $\mathbb{Z}_p$-invariant. As remarked in Definition 3.4, the collection of open sets appearing in the intersection (3.5) is closed under finite intersection. In particular, we could consider just those U contained in $B(1)$. Thus *a fortiori* X^+ is the intersection of all open sets U with $X \subseteq U \subseteq M$ and $H_2(U) = 0$. The set of such U is permuted by $\mathbb{Z}_p$, so X^+ is $\mathbb{Z}_p$-invariant. $\square$

4.3 Step 3: Construction of an interesting compact set $Z \subseteq M$

In this section, we construct a compact $\mathbb{Z}_p$-invariant set $Z \subseteq M$ such that:

1. On a coarse scale, Z looks like a handlebody of genus two.
2. The action of $\mathbb{Z}_p$ on $\check{H}^1(Z)$ is nontrivial.

We follow a natural strategy to construct the set Z. We first let K be the orbit under $\mathbb{Z}_p$ of a closed handlebody of genus two, and we then construct a set L as the orbit under $\mathbb{Z}_p$ of a small arc connecting two points on the boundary of K. Then we let $Z = K \cup L$. This strategy is complicated by the fact that the orbit set K is *a priori* very wild, and we will need to investigate the connectedness properties of it and other sets in order to make the construction work. We will take advantage of the tools we have developed in Section 3.3 and the setup from Sections 4.1–4.2. Even with this preparation, though, careful verification of each step takes us a while.

Let $x_0 \in B(\eta) \setminus \operatorname{Fix} \mathbb{Z}_p$ (such an x_0 exists by Step 1(iii)).

Definition 4.3 (see Figure 1)**.** Let $K_0 \subseteq B(1)$ be a closed handlebody of genus two whose unique point of lowest z-coordinate is x_0. Let $K = (\mathbb{Z}_p K_0)^+$.

We think of K as being illustrated essentially by Figure 1; it's just K_0 plus some wild fuzz of width η. Note that K is compact, $\mathbb{Z}_p$-invariant, and $K = K^+$.

Lemma 4.4. *If $A \subseteq \partial K$ is any finite set, then $K \setminus \mathbb{Z}_p A$ is path-connected.*

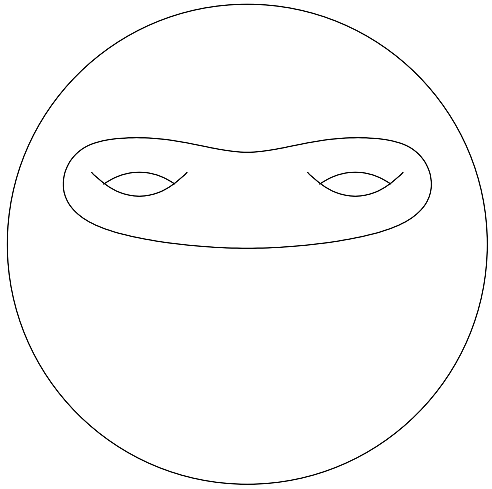

Figure 1: Diagram of K_0 inside $B(1) \subseteq M$.

Proof. By definition, K_0° is path-connected. Now the action of $\mathbb{Z}_p$ is within $\eta = 2^{-10}$ of the identity, so $\mathbb{Z}_p(K_0^\circ)$ is also path-connected.

Now suppose $x \in \mathbb{Z}_p K_0 \setminus \mathbb{Z}_p A$. Then there exists $\alpha \in \mathbb{Z}_p$ so that $\alpha^{-1}x \in K_0$. From $\alpha^{-1}x$ there is clearly a path inwards to K_0°, and this is disjoint from $\mathbb{Z}_p A$ since $\mathbb{Z}_p A \subseteq \partial K$. Translating this path by α shows that $\mathbb{Z}_p K_0 \setminus \mathbb{Z}_p A$ is path-connected.

Now suppose $x \in K \setminus \mathbb{Z}_p A$. If $x \notin \mathbb{Z}_p K_0$, then let V denote the (open) connected component of $K \setminus \mathbb{Z}_p K_0$ containing x. By Remark 3.7 and Lemma 3.8, we can find a path from x to infinity in $\mathbb{R}^3 \setminus \mathbb{Z}_p A$. There is a time when this path first hits $\partial V \subseteq \mathbb{Z}_p K_0$, thus giving us a path from x to $\mathbb{Z}_p K_0$ contained in $K \setminus \mathbb{Z}_p A$. Thus $K \setminus \mathbb{Z}_p A$ is path-connected, as was to be shown. $\square$

Lemma 4.5. *There exists a compact $\mathbb{Z}_p$-invariant set $L \subseteq M$ of diameter $\leq 4\eta$ so that $L = L^+$ and $K \cap L = \mathbb{Z}_p x_1 \cup \mathbb{Z}_p x_2$ where $x_1 \notin \operatorname{Fix} \mathbb{Z}_p$, $\mathbb{Z}_p x_1 \cap \mathbb{Z}_p x_2 = \varnothing$, and L contains a path from x_1 to x_2.*

Proof. Let $x_1 \in K$ be any point of lowest z-coordinate in K. We claim that $x_1 \notin \operatorname{Fix} \mathbb{Z}_p$. First, observe that since $x_1 \in K = (\mathbb{Z}_p K_0)^+$ is a point of lowest z-coordinate, necessarily $x_1 \in \mathbb{Z}_p K_0$. If $x_1 \in K_0$, then $x_1 = x_0$ (recall from Definition 4.3 that x_0 is the point of lowest z-coordinate in K_0), and by definition $x_0 \notin \operatorname{Fix} \mathbb{Z}_p$. On the other hand, if $x_1 \in (\mathbb{Z}_p K_0) \setminus K_0$, then certainly $x_1 \notin \operatorname{Fix} \mathbb{Z}_p$. Thus $x_1 \notin \operatorname{Fix} \mathbb{Z}_p$.

Since $x_1 \in \mathbb{Z}_p K_0$, we may pick $x_2' \in \mathbb{Z}_p(K_0^\circ) \subseteq K^\circ$ which is arbitrarily close to x_1. Specifically, let us fix $x_2' \in K^\circ \cap B(x_1, \eta)$. Note that $x_2' \notin \mathbb{Z}_p x_1$ since $x_1 \notin K^\circ$. Now we claim that there exists a continuous path $\gamma : [0,1] \to B(x_1, \eta)$ such that:

i. $\gamma(0) = x_1$.

ii. $\gamma(1) = x_2'$.

iii. $\gamma^{-1}(\mathbb{Z}_p x_1) = \{0\}$.

iv. 0 is not a limit point of $\gamma^{-1}(K)$.

To construct such a path γ, argue as follows. First, define $\gamma : [0, \frac{1}{2}] \to B(x_1, \eta)$ to be some path straight downward from x_1 (this takes care of (i)). Since x_1 is a point of smallest z-coordinate in $K \supseteq \mathbb{Z}_p x_1$, this also takes care of (iv) and is consistent with (iii). Now by

Remark 3.7 and Lemma 3.8, we know $B(x_1, \eta) \setminus \mathbb{Z}_p x_1$ is path-connected, so we can find a path $\gamma : [\frac{1}{2}, 1] \to B(x_1, \eta) \setminus \mathbb{Z}_p x_1$ from $\gamma(\frac{1}{2})$ to x_2' (this takes care of (ii) and (iii)). Splicing these two functions together gives $\gamma : [0, 1] \to B(x_1, \eta)$ satisfying (i), (ii), (iii), and (iv).

Now by (iv) we have $t := \min(\gamma^{-1}(K) \setminus \{0\})$ exists and is positive. Let $x_2 = \gamma(t)$, and define $L_0 = \gamma([0, t])$. By (iii), we have $\mathbb{Z}_p x_1 \cap \mathbb{Z}_p x_2 = \varnothing$. Define $L = (\mathbb{Z}_p L_0)^+$. By construction, L contains a path from x_1 to x_2. Certainly $L_0 \subseteq B(x_1, \eta)$ so $L \subseteq B(x_1, 2\eta)$ by Step 1(ii), and so L has diameter $\leq 4\eta$. It remains only to show that $K \cap L = \mathbb{Z}_p x_1 \cup \mathbb{Z}_p x_2$ (certainly the containment $\supseteq$ is given by definition).

Note that $K \setminus \mathbb{Z}_p L_0 = K \setminus (\mathbb{Z}_p x_1 \cup \mathbb{Z}_p x_2)$, which by Lemma 4.4 is path-connected. Thus $K \setminus \mathbb{Z}_p L_0$ lies in a single connected component of $\mathbb{R}^3 \setminus \mathbb{Z}_p L_0$. Now $\mathbb{Z}_p L_0$ has diameter $\leq 4\eta$ as remarked above, and $K \setminus \mathbb{Z}_p L_0$ contains a large handlebody, so it must lie in the unbounded component of $\mathbb{R}^3 \setminus \mathbb{Z}_p L_0$. Hence $K \setminus (\mathbb{Z}_p x_1 \cup \mathbb{Z}_p x_2)$ is disjoint from $(\mathbb{Z}_p L_0)^+ = L$. ☐

Definition 4.6. Let $Z = K \cup L$.

4.4 Step 4: The cohomology of Z

Lemma 4.7. *$Z = Z^+$, and the action of $\mathbb{Z}_p$ on $\check{H}^1(Z)$ is nontrivial.*

Proof. The key to proving both statements is the following Mayer–Vietoris sequence (which exists and is exact by Lemma 3.2):

$$\cdots \to \check{H}^*(Z) \to \check{H}^*(K) \oplus \check{H}^*(L) \to \check{H}^*(K \cap L) \to \check{H}^{*+1}(Z) \to \cdots \tag{4.2}$$

By Lemma 4.5, $K \cap L = \mathbb{Z}_p x_1 \cup \mathbb{Z}_p x_2$, so by Remark 3.7, $\check{H}^i(K \cap L) = 0$ for $i > 0$. Thus we have $\check{H}^2(Z) = \check{H}^2(K) \oplus \check{H}^2(L)$. Now applying Lemma 3.5(iv) twice shows that $K = K^+$ and $L = L^+$ together imply that $Z = Z^+$.

Now let us show that $\mathbb{Z}_p$ acts on $\check{H}^1(Z)$ nontrivially. We use the exact sequence:

$$\check{H}^0(K) \oplus \check{H}^0(L) \to \check{H}^0(K \cap L) \to \check{H}^1(Z) \tag{4.3}$$

Note that $\check{H}^0$ is just the group of locally constant functions to $\mathbb{Z}$. Thus it suffices to exhibit a locally constant function $q : K \cap L \to \mathbb{Z}$ and an element $\alpha \in \mathbb{Z}_p$ such that $\alpha q - q$ is not in the image of $\check{H}^0(K) \oplus \check{H}^0(L)$. Let us define:

$$q(r) = \begin{cases} 1 & r \in p\mathbb{Z}_p x_1 \\ 0 & r \in \bigcup_{a \in (\mathbb{Z}/p) \setminus \{0\}} (a + p\mathbb{Z}_p) x_1 \cup \mathbb{Z}_p x_2 \end{cases} \tag{4.4}$$

(observe by Lemma 4.5 that $x_1 \notin \operatorname{Fix} \mathbb{Z}_p$, so the sets on the right hand side are indeed disjoint). Let $\alpha \in \mathbb{Z}_p$ be congruent to 1 mod p. Then we have:

$$(\alpha q - q)(r) = \begin{cases} -1 & r \in p\mathbb{Z}_p x_1 \\ 1 & r \in (1 + p\mathbb{Z}_p) x_1 \\ 0 & r \in \bigcup_{a \in (\mathbb{Z}/p) \setminus \{0,1\}} (a + p\mathbb{Z}_p) x_1 \cup \mathbb{Z}_p x_2 \end{cases} \tag{4.5}$$

Now by Lemma 4.5, x_1, x_2 are in the same component of L, and by Lemma 4.4 (taking $A = \varnothing$), they are in the same component of K. Thus every function in the image of $\check{H}^0(L) \oplus \check{H}^0(K)$ assigns the same value to x_1 and x_2. Clearly this is not the case for $\alpha q - q$, so we are done. ☐

We just proved that $Z = Z^+$, so by Lemma 3.6, $N_\epsilon^{\mathrm{inv}}(Z)^+$ is a final system of neighborhoods of Z. Thus by Lemma 3.1 we have $\check{H}^1(Z) = \varinjlim H^1(N_\epsilon^{\mathrm{inv}}(Z)^+)$ as abelian groups with an action of $\mathbb{Z}_p$. Since the action on the limit group is nontrivial, the following definition makes sense.

Definition 4.8. Fix $\epsilon \in (0, \eta)$ such that the $\mathbb{Z}_p$ action on the image of $H^1(N_\epsilon^{\mathrm{inv}}(Z)^+) \to \check{H}^1(Z)$ is nontrivial. Define $U = N_\epsilon^{\mathrm{inv}}(Z)^+ \setminus Z$.

4.5 Step 5: Special elements of $\mathcal{S}(U)$

Lemma 4.9. *The set U is an approximate surface in the sense of Definition 2.3.*

Proof. We have $\mathbb{R}^3 \setminus U = Z \cup (\mathbb{R}^3 \setminus N_\epsilon^{\mathrm{inv}}(Z)^+)$; the latter two sets are connected and disjoint, so by Lemma 3.3, we have $H_2(U) = \mathbb{Z}$. The existence of a nontrivial embedded sphere in U would imply that the map $H^1(N_\epsilon^{\mathrm{inv}}(Z)^+) \to \check{H}^1(Z)$ is trivial, contradicting the definition of U; thus U is irreducible. Certainly U has at least two ends, and since $H_2(U) = \mathbb{Z}$ it has at most two ends. Since U is an open subset of $\mathbb{R}^3$, it is orientable. $\square$

Recall the definition of $(\mathcal{S}(U), \leq)$ from Section 2. Certainly the action of $\mathbb{Z}_p$ on U induces an action on $\mathcal{S}_{\mathrm{TOP}}(U)$.

Lemma 4.10. *There exists an element $\mathfrak{F} \in \mathcal{S}(U)$ which is fixed by $\mathbb{Z}_p$.*

Proof. Observe that if F is any surface in U, then for sufficiently large k, we have that αF is homotopic to F for all $\alpha \in p^k\mathbb{Z}_p$. Thus $\mathbb{Z}_p$ acts on $\mathcal{S}(U)$ with finite orbits. Now any group acting on a nonempty lattice with finite orbits has a fixed point, namely the least upper bound of any orbit. Recall that $\mathcal{S}(U)$ is nonempty by Remark 2.7. $\square$

Fix a PL π_1-injective surface $F \subseteq U$ such that $[F] \in \mathcal{S}(U)$ is fixed by $\mathbb{Z}_p$. Denote by $\mathrm{int}(F)$ and $\mathrm{ext}(F)$ the two connected components of $\mathbb{R}^3 \setminus F$.

Lemma 4.11. *The $\mathbb{Z}_p$ action on U induces a homomorphism $\mathbb{Z}_p \to \mathrm{MCG}(F)$, as well as actions on all the homology and cohomology groups appearing in (4.6)–(4.9). These actions are compatible with the maps in (4.6)–(4.9), as well as with the map $\mathrm{MCG}(F) \to \mathrm{Aut}(H_1(F))$.*

$$\begin{tikzcd} H^1(N_\epsilon^{\mathrm{inv}}(Z)^+) \arrow[r] & H^1(\mathrm{int}(F)) \arrow[r] \arrow[d] & \check{H}^1(Z) \\ & H^1(F) & \end{tikzcd} \tag{4.6}$$

$$H_1(Z) \to H_1(\mathrm{int}(F)) \tag{4.7}$$

$$H_1(M \setminus N_\epsilon^{\mathrm{inv}}(Z)^+) \to H_1(\mathrm{ext}(F)) \tag{4.8}$$

$$H_1(F) \xrightarrow{\sim} H_1(\mathrm{int}(F)) \oplus H_1(\mathrm{ext}(F)) \tag{4.9}$$

Proof. For every $\alpha \in \mathbb{Z}_p$, we know by Lemma 2.2 that there is a compactly supported isotopy of U sending αF to F. Now such an isotopy can clearly be extended to all of M as the constant isotopy on $M \setminus U$. Thus for every $\alpha \in \mathbb{Z}_p$, let us pick an isotopy:

$$\psi_\alpha^t : M \to M \qquad (t \in [0,1]) \tag{4.10}$$

so that $\psi_\alpha^0 = \mathrm{id}_M$, $\psi_\alpha^1(\alpha F) = F$, and $\psi_\alpha^t(x) = x$ for $x \in M \setminus U$. Denote by $T_\alpha : M \to M$ the action of $\alpha \in \mathbb{Z}_p$.

Now observe that for every $\alpha \in \mathbb{Z}_p$, the map $\psi_\alpha^1 \circ T_\alpha$ fixes Z, $N_\epsilon^{\mathrm{inv}}(Z)^+$, and F. Thus the map $\psi_\alpha^1 \circ T_\alpha$ induces an automorphism of each of the diagrams (4.6)–(4.9) and gives a compatible element of $\mathrm{MCG}(F)$. It remains only to show that this is a homomorphism from $\mathbb{Z}_p$.

We need to show that for all $\alpha, \beta \in \mathbb{Z}_p$, the two maps $\psi_\beta^1 \circ T_\beta \circ \psi_\alpha^1 \circ T_\alpha$ and $\psi_{\alpha+\beta}^1 \circ T_{\alpha+\beta}$ induce the same action on (4.6)–(4.9) and give the same element of $\mathrm{MCG}(F)$ (we use additive notation for the group law of $\mathbb{Z}_p$, though the fact that $\mathbb{Z}_p$ is abelian is irrelevant). It of course suffices to show that $\psi_\beta^1 \circ T_\beta \circ \psi_\alpha^1 \circ T_\alpha \circ (\psi_{\alpha+\beta}^1 \circ T_{\alpha+\beta})^{-1}$ induces the trivial action on (4.6)–(4.9) and gives the trivial element of $\mathrm{MCG}(F)$. Now we write:

$$\begin{aligned} \psi_\beta^1 \circ T_\beta \circ \psi_\alpha^1 \circ T_\alpha \circ (\psi_{\alpha+\beta}^1 \circ T_{\alpha+\beta})^{-1} &= \psi_\beta^1 \circ T_\beta \circ \psi_\alpha^1 \circ T_\alpha \circ T_{\alpha+\beta}^{-1} \circ (\psi_{\alpha+\beta}^1)^{-1} \\ &= \psi_\beta^1 \circ T_\beta \circ \psi_\alpha^1 \circ T_\beta^{-1} \circ (\psi_{\alpha+\beta}^1)^{-1} \end{aligned} \tag{4.11}$$

This is the identity map on Z and on $M \setminus N_\epsilon^{\mathrm{inv}}(Z)^+$, so the action on their homology is trivial. The map (4.11) is isotopic to the identity map via $\psi_\beta^t \circ T_\beta \circ \psi_\alpha^t \circ T_\beta^{-1} \circ (\psi_{\alpha+\beta}^t)^{-1}$ for $t \in [0,1]$, which only moves points in a compact subset of U. Thus the action on the homology of $N_\epsilon^{\mathrm{inv}}(Z)^+$ is trivial as well. By Lemma 2.12, this isotopy induces the trivial element in $\mathrm{MCG}(F)$, and this means that the action on the homology and cohomology of F, $\mathrm{int}(F)$, and $\mathrm{ext}(F)$ is trivial as well. □

Lemma 4.12. *The map $\mathbb{Z}_p \to \mathrm{MCG}(F)$ annihilates an open subgroup of $\mathbb{Z}_p$ and has nontrivial image.*

Proof. The action of $\mathbb{Z}_p$ on U is continuous, so for sufficiently large k, we have that αF and F are homotopic as maps $F \to U$ for all $\alpha \in p^k\mathbb{Z}_p$. Thus the homomorphism $\mathbb{Z}_p \to \mathrm{MCG}(F)$ annihilates a neighborhood of the identity in $\mathbb{Z}_p$.

To prove that the image is nontrivial, it suffices to show that the $\mathbb{Z}_p$ action on $H^1(F)$ is nontrivial. For this, consider equation (4.6). By Definition 4.8, there exists an element of $H^1(N_\epsilon^{\mathrm{inv}}(Z)^+)$ whose image in $\check{H}^1(Z)$ is not fixed by $\mathbb{Z}_p$. Thus the action of $\mathbb{Z}_p$ on $H^1(\mathrm{int}(F))$ is nontrivial. Now the vertical map $H^1(\mathrm{int}(F)) \to H^1(F)$ is injective, so the action of $\mathbb{Z}_p$ on $H^1(F)$ is nontrivial as well. □

Lemma 4.13. *There is a rank four submodule of $H_1(F)^{\mathbb{Z}_p}$ on which the intersection form is:*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.12}$$

Proof. There are two obvious loops in K_0 (Definition 4.3) generating its homology, and two obvious dual loops in $M \setminus N_\epsilon^{\mathrm{inv}}(Z)^+$. Since the $\mathbb{Z}_p$ action is within $\eta = 2^{-10}$ of the identity, it is easy to see that their classes in homology are fixed by $\mathbb{Z}_p$. Thus we have well-defined classes $\alpha_1, \alpha_2 \in H_1(Z)^{\mathbb{Z}_p}$ and $\beta_1, \beta_2 \in H_1(M \setminus N_\epsilon^{\mathrm{inv}}(Z)^+)^{\mathbb{Z}_p}$ with linking numbers $\mathrm{lk}(\alpha_i, \beta_j) = \delta_{ij}$. Now the maps from equations (4.7), (4.8), and the isomorphism (4.9) give us corresponding elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in H_1(F)^{\mathbb{Z}_p}$. The intersection form on $H_1(F)$ coincides with the linking form between $H_1(\mathrm{int}(F))$ and $H_1(\mathrm{ext}(F))$. Thus the intersection form on the submodule of $H_1(F)^{\mathbb{Z}_p}$ generated by $\alpha_1, \alpha_2, \beta_1, \beta_2$ is indeed given by (4.12). □

By Lemma 4.12, the image of $\mathbb{Z}_p$ in $\mathrm{MCG}(F)$ is a nontrivial cyclic p-group. It has a (unique) subgroup isomorphic to $\mathbb{Z}/p$, and by Lemma 4.13 this subgroup $\mathbb{Z}/p \subseteq \mathrm{MCG}(F)$ has the property that $H_1(F)^{\mathbb{Z}/p}$ has a submodule on which the intersection form is given by (4.12). This contradicts Lemma 5.1 below, and thus completes the proof of Theorem 1.5. □

Remark 4.14. One expects that if we make U thick enough around most of Z (but still thin near L), then away from L the surface F can be made to look like a punctured surface of genus two. However, it is not clear how to prove this stronger, more geometric property about F.

If we could prove this, then the final analysis is more robust. Instead of homology classes $\alpha_1, \alpha_2, \beta_1, \beta_2 \in H_1(F)^{\mathbb{Z}_p}$, we now have simple closed curves $\alpha_1, \alpha_2, \beta_1, \beta_2$ in F which are fixed up to isotopy by the $\mathbb{Z}_p$ action. Now the finite image of $\mathbb{Z}_p$ in $\mathrm{MCG}(F)$ is realized as a subgroup of $\mathrm{Isom}(F, g)$ for some hyperbolic metric g on F. A hyperbolic isometry fixing $\alpha_1, \alpha_2, \beta_1, \beta_2$ fixes their unique length-minimizing geodesic representatives, and thus fixes their intersections, forcing the isometry to be the identity map. This contradicts the nontriviality of the homomorphism $\mathbb{Z}_p \to \mathrm{MCG}(F)$. As the reader will readily observe, it is enough that α_1, β_1 be fixed, so we could let K_0 be a handlebody of genus one and this argument would go through.

5 Actions of $\mathbb{Z}/p$ on closed surfaces

Lemma 5.1. *Let F be an oriented closed surface, and let $\mathbb{Z}/p \subseteq \mathrm{MCG}(F)$ be some cyclic subgroup of prime order. Then the intersection form of F restricted to $H_1(F)^{\mathbb{Z}/p}$ is a direct sum of some number of copies of $\left(\begin{smallmatrix} 0 & p \\ -p & 0 \end{smallmatrix}\right)$ and at most one copy of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ (which is present if and only if the action is free). In particular, modulo p it has rank at most two.*

Proof. One classifies all subgroups $\mathbb{Z}/p \subseteq \mathrm{MCG}(F)$ using results of Nielsen as follows. First, Nielsen [24] showed that given any cyclic subgroup $G \subseteq \mathrm{MCG}(F)$ (not necessarily of prime order), there exists a constant curvature metric g on F such that $G \subseteq \mathrm{Isom}(F, g) \subseteq \mathrm{MCG}(F)$ (see Thurston [40] or Kerckhoff [14] for a modern perspective on this fact), so in particular, $G \subseteq \mathrm{MCG}(F)$ is realized by a genuine action of G on F. Second, Nielsen [23] showed that actions of finite cyclic groups on closed surfaces are classifed by their "fixed point data". These two statements allow an explicit construction of all subgroups $\mathbb{Z}/p \subseteq \mathrm{MCG}(F)$.

A recent paper by Chen–Glover–Jensen [4] presents the construction, and the present lemma follows immediately. If one restricts to p odd, then Symonds [38, pp389–390 Theorem B] goes further and works out the action on $H^1(F)$; one can immediately read off the present lemma using Poincaré duality. □

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